ON A RISK MODEL WITH A CONSTANT DIVIDEND AND DEBIT INTEREST

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Abstract

In this paper, we consider a risk model with debit interest and a constant dividend barrier. We assume that the claim number process is the generalized Erlang (n) process. For this model, the moment-generation function, the moments of the discounted dividend payments, and the Gerber-Shiu discounted penalty function are investigated. Integral equations and integro-differential equations with certain boundary conditions for them are derived.

1. Introduction

In recent years, the issue of absolute ruin has received remarkable attention in the actuarial literature. We assume that, when the surplus is negative or the insurer is on deficit, the insurer could borrow an amount of money equal to the deficit at a debit interest force $\delta > 0$. Meanwhile, the insurer will repay the debts continuous from her premium income. Thus, the surplus of the insurer is driven under the debit interest force $\delta$,
when the surplus is negative. The negative surplus may return to a positive level. However, when the negative surplus attains the level \(-c/\delta\) or is below \(-c/\delta\), the surplus is no longer able to be positive, because the debts of the insurer at this time are greater than or equal to \(-c/\delta\), which is the present value at that time for all premium income available after that point. Absolute ruin occurs at this moment.

The absolute ruin probability has been studied by Gerber [7] for the compound Poisson model, when the debit and the credit interest rates are the same. A closed form solution is given in the case of an exponential claim amount distribution. Dassios and Embrechts [3] considered the ruin probability with exponential claims by using the martingale approach, and the theory of piecewise deterministic Markov processes. Embrechts and Schmidili [5] considered the absolute ruin probability for a more complicated risk model. They assumed that, the company can borrow money when the surplus is negative, and receive interest for capital above a certain level. Dickson and Egidio dos Reis [4] and Zhang and Wu [11] discussed the effect of interest on the negative surplus. Cai [2] considered the Gerber-Shiu discounted penalty function at absolute ruin. Wang and Yin [9] discussed the moment generation function and the moments of the discounted dividend payments. Gao and Yin [6] considered the moment-generation function in a Sparre Andersen model perturbed by diffusion with generalized Erlang(n)-distributed inter-claim times and a threshold dividend strategy.

Motivated by Cai [2] and Gao and Yin [6], in this paper, we study the expected penalty function, the moment-generation function, and the moments of the discounted dividend payments at absolute ruin.

Let \((\Omega, \mathbf{F}, \mathbf{P})\) be a complete probability space containing all random objects defined in the following. Suppose that the surplus of an insurer follows the renewal risk process

\[
U(t) = u + ct - \sum_{k=1}^{N(t)} Z_k = u + ct - S(t), \quad t \geq 0,
\]

where \(u\) and \(c\) are constants, \(u\) is the initial surplus, and \(c > 0\) is the rate of premium, \(S(t) = \sum_{k=1}^{N(t)} Z_k\) is the aggregate claims process, \(\{Z_k : k =\)
is a sequence of independent, and identically distributed claim
amount nonnegative random variables with a common distribution
function $F$. The ordinary renewal process $\{N(t), t \geq 0\}$ denotes the
number of claims up to time $t$, with $N(t) = \max\{k \geq 1 : W_1 + W_2 + \cdots + W_k \leq t\}$, where the i.i.d claim waiting times $W_i$ have a common
generalized Erlang$(n)$ distribution, i.e., the $W_i$’s are distributed as the
sum of $n$ independent and exponentially distributed random variables:

$$W_i = V_1 + V_2 + \cdots + V_n, \quad i = 1, 2, \cdots, n,$$

where $V_i$ may have different exponential parameters $\beta_i > 0$.
Furthermore, we assume that $\{W_i\}_{i \geq 1}$ and $\{Z_i\}_{i \geq 1}$ are independent and $cE(W_1) > E(X_1)$.

We now consider a threshold dividend strategy, dividends are paid at
a constant rate $\alpha$ with $\alpha < c$ whenever $U_b(t) \geq b(b > 0)$, and no dividends are paid whenever $U_b(t) < b$. Let $\{U_b(t), t \geq 0\}$ be the
modified surplus of the insurer at time $t$. Then it can be expressed as

$$dU_b(t) = \begin{cases} (c - \alpha)dt - dS(t), & \text{if } U_b(t) \geq b; \\ cdt - dS(t), & \text{if } b > U_b(t) \geq 0; \quad t \geq 0, \\ (c + \delta U_b(t))dt - dS(t), & \text{if } 0 > U_b(t) \geq -\frac{c}{\delta}. \end{cases}$$

where $b(> 0)$ denotes the amount of capital the company retains as a
liquid reserve, $a(> c)$ is the dividend rate, and $\delta(> 0)$ denotes the force of
interest for borrowed money.

Let $T_\delta = \inf\{t \geq 0 : U_b(t) \leq -\frac{c}{\delta}\}$ denote the absolute ruin time of
the surplus process $\{U_b(t), t \geq 0\}$, where $T_\delta = \infty$, if absolute ruin does
not occur in any finite time. We define the Gerber-Shiu expected
discounted penalty function at absolute ruin by

$$\Phi(u) = E(e^{-\gamma T_\delta} \omega(U_{T_\delta}; |U_{T_\delta}|)I(T_\delta < \infty) | U(0) = u),$$

(1.3)
where \( w(x_1, x_2) \in \left[ -\frac{c}{\delta}, \infty \right) \times \left[ -\frac{c}{\delta}, \infty \right) \) is a nonnegative function, which denotes the penalty due at absolute ruin, \( U_{T_\delta} \) is the surplus immediately prior to absolute ruin time, and \( |U_{T_\delta}| \) is the deficit at absolute ruin time, \( \gamma \) is a positive constant, which can be viewed as the interest force for the calculation of the present value of the penalty, and \( I(A) \) is the indicator function of an event \( A \). In particular, the absolute ruin probability is denoted by

\[
\Psi(u) = P(T_\delta < \infty | U(0) = u).
\]

Let \( D(t) \) be the aggregate dividends paid from 0 to \( t \). Let \( D_{u,b} = \int_0^{T_\delta} e^{-\lambda s} dD(s) \) be the present value of all dividends until absolute ruin, where \( \lambda \) is the discount factor. We denote the moment-generating function of \( D_{u,b} \) by \( M(u, y, b) = E[e^{\gamma D_{u,b}}] \).

In this paper, we will study the expected discounted penalty function, the moment-generation function, and the moments of the discounted dividend payments at absolute ruin. In Section 2, we get integro-differential equations and integral equations of the expected discounted penalty function at absolute ruin. In Section 3, we get the integro-differential equations of the moment-generation function and the moments of the discounted dividend payments at absolute ruin.

### 2. The Gerber-Shiu Discounted Penalty Function

Since the Gerber-Shiu function \( \Phi(u) \) has different paths for \( -\frac{c}{\delta} \leq u < 0, 0 \leq u \leq b, \) and \( u > b \), we define

\[
\Phi(u) = \begin{cases} 
\Phi_1(u), & b > u \geq 0, \\
\Phi_2(u), & u \geq b, \\
\Phi_3(u), & 0 > u \geq -\frac{c}{\delta}.
\end{cases}
\]

Similarly, we let
The following theorem provides integro-differential equations for the Gerber-Shiu function $\Phi(u)$.

**Theorem 2.1.** The Gerber-Shiu function $\Phi(u)$ satisfies the following integro-differential equations:

When $0 < u < b$, we have

$$
\sum_{i=1}^{n} \left[ (1 + \frac{\gamma_i}{\beta_i}) I - \frac{c - \delta}{\beta_i} \frac{\partial}{\partial u} \right] \Phi_1(u) = \int_{0}^{u} \Phi_1(u - z) dF(z) + \int_{u}^{u+b} \Phi_2(u - z) dF(z) + A(u), \quad (2.1)
$$

when $u > b$, we have

$$
\sum_{i=1}^{n} \left[ (1 + \frac{\gamma_i}{\beta_i}) I - \frac{c - \delta}{\beta_i} \frac{\partial}{\partial u} \right] \Phi_2(u) = \int_{u-b}^{u} \Phi_1(u - z) dF(z) + \int_{u}^{u+b} \Phi_3(u - z) dF(z) + A(u), \quad (2.2)
$$

and when $-\frac{c}{\delta} < u < 0$,

$$
\sum_{i=1}^{n} \left[ (1 + \frac{\gamma_i}{\beta_i}) I - \frac{c + \delta u}{\beta_i} \frac{\partial}{\partial u} \right] \Phi_3(u) = \int_{0}^{u+\delta} \Phi_3(u - z) dF(z) + A(u), \quad (2.3)
$$

with the boundary conditions for $k = 0, 1, 2, \ldots, n-1$,

$$
\left. \sum_{i=1}^{k} \left[ (1 + \frac{\gamma_i}{\beta_i}) I - \frac{c + \delta u}{\beta_i} \frac{\partial}{\partial u} \right] \Phi_3(u) \right|_{u=-\frac{c}{\delta}+0} = A(-\frac{c}{\delta}), \quad (2.4)
$$
\[
\prod_{i=1}^{k} \left[ (1 + \frac{\gamma}{\beta_i})I - \frac{c}{\beta_i} \frac{\partial}{\partial u} \right] \Phi_1(u)|_{u=b,-} = \prod_{i=1}^{k} \left[ (1 + \frac{\gamma}{\beta_i})I - \frac{c - \alpha}{\beta_i} \frac{\partial}{\partial u} \right] \Phi_2(u)|_{u=b},
\]

(2.5)

\[
\prod_{i=1}^{k} \left[ (1 + \frac{\gamma}{\beta_i})I - \frac{c}{\beta_i} \frac{\partial}{\partial u} \right] \Phi_1(u)|_{u=0,+} = \prod_{i=1}^{k} \left[ (1 + \frac{\gamma}{\beta_i})I - \frac{c + \delta u}{\beta_i} \frac{\partial}{\partial u} \right] \Phi_3(u)|_{u=0,-},
\]

(2.6)

where \( A(u) = \int_{u+\frac{\xi}{\delta}}^{\infty} \omega(u, u - z) dF(z), \prod_{i=1}^{n} = 1 \) and I is the identity operator.

**Proof.** For notional convenience, we let \( \Phi_{ik}(u) (i = 1, 2, 3; k = 1, 2, \ldots, n) \) denote the Gerber-Shiu function, when the distribution of the interclaim is generalized Erlang \((k)\) process and \( \Phi_i(u) = \Phi_{in}(u) (i = 1, 2, 3) \).

By conditioning on the time and amount of the first claim and discounting, the expected values to time 0 at the interest force \( \gamma \), when \( 0 \leq u \leq b \), we obtain

\[
\Phi_{1n}(u) = \int_{0}^{t_b} e^{-\gamma t} k_n(t) \left[ \int_{0}^{u+ct} \Phi_{1n}(u + ct - y) dF(y) \right] dt + \int_{u+ct}^{u+ct+\frac{\xi}{\delta}} \Phi_{3n}(u + ct - y) dF(y) + \int_{u+ct+\frac{\xi}{\delta}}^{\infty} w(u + ct, y - c + u) dF(y) \right] dt
\]

\[
+ \int_{0}^{\infty} e^{-\gamma t} k_n(t) \left[ \int_{0}^{b+(c-\alpha)(t-t_b)} \Phi_{1n}(b + (c - \alpha)(t - t_b) - y) dF(y) \right] dt + \int_{b+(c-\alpha)(t-t_b)+\frac{\xi}{\delta}}^{\infty} \Phi_{2n}(b + (c - \alpha)(t - t_b) - y) dF(y) \right] dt
\]

\[
+ \int_{b+(c-\alpha)(t-t_b)+\frac{\xi}{\delta}}^{\infty} w(b + (c - \alpha)(t - t_b), y - (b + (c - \alpha)(t - t_b))) dF(y) \right] dt,
\]
where \( t_b = \frac{b - u}{c} \) and \( k_n(t) \) is the common density function of the claim waiting times \( W_i \)'s. When \( u > b \), we have

\[
\Phi_{2n}(u) = \int_0^{+\infty} e^{-yt} k_n(t) \left[ \int_0^{u+(c-\alpha)t-b} \Phi_{2n}(u+(c-\alpha)t-y) dF(y) \right] + \int_{u+(c-\alpha)t}^{+\infty} \Phi_{1n}(u+(c-\alpha)t-y) dF(y) \\
+ \int_{u+(c-\alpha)t}^{+\infty} \Phi_{3n}(u+(c-\alpha)t-y) dF(y) \\
+ \int_{u+(c-\alpha)t}^{+\infty} w(u+(c-\alpha)t, y-(u+(c-\alpha)t)) dF(y) dt.
\]

When \( 0 > u > -\frac{c}{\delta} \), we have

\[
\Phi_{3n}(u) = \int_0^{+\infty} e^{-yt} k_n(t) \left[ \int_0^{h_{\delta}(t, u)+\frac{\xi}{\delta}} \Phi_{3n}(h_{\delta}(t, u)-y) dF(y) \right] + \int_{h_{\delta}(t, u)+\frac{\xi}{\delta}}^{+\infty} w(h_{\delta}(t, u), y-h_{\delta}(t, u)) dF(y) dt \\
+ \int_{h_{\delta}(t, u)+\frac{\xi}{\delta}}^{+\infty} e^{-rt} k_n(t) \left[ \int_0^{c(t-t_0)} \Phi_{1n}(c(t-t_0)-y) dF(y) \right] + \int_{c(t-t_0)}^{+\infty} \Phi_{3n}(c(t-t_0)-y) dF(y) + \int_{c(t-t_0)+\frac{\xi}{\delta}}^{+\infty} w(c(t-t_0), y-c(t-t_0)) dF(y) dt.
\]

First changing variable \( x = u + ct \) with respect to \( t \) from 0 to \( t_b(t_b = \frac{b - u}{c} \) ), and then changing variable \( z = b + (c - \alpha)(t - t_b) \) with respect to \( t \) from \( t_b \) to \( +\infty \) in \( \Phi_{1n}(u) \), we obtain
\[
\Phi_{1n}(u) = \int_{u}^{b} e^{-\frac{x-u}{c}} k_n\left(\frac{x-u}{c}\right) \frac{1}{c} \int_{0}^{x} \Phi_{1n}(x-y)dF(y)
\]
\[
+ \int_{x}^{x+\frac{\xi}{\delta}} \Phi_{3n}(x-y)dF(y) + \int_{x+\frac{\xi}{\delta}}^{+\infty} u(x, y-x)dF(y)dx
\]
\[
+ \int_{b}^{+\infty} e^{-\frac{y-z}{c-\alpha}} \frac{b-u}{c} k_n\left(\frac{z-b}{c-\alpha} + \frac{b-u}{c}\right) \int_{0}^{z} \Phi_{1n}(z-y)dF(y)
\]
\[
+ \int_{z}^{z+\frac{\xi}{\delta}} \Phi_{3n}(z-y)dF(y) + \int_{z+\frac{\xi}{\delta}}^{+\infty} w(z, y-z)dF(y) \right] \frac{1}{c-\alpha} dz.
\]

If we let \( u + (c-\alpha)t = x \), then \( \Phi_{2n}(u) \) becomes
\[
\Phi_{2n}(u) = \int_{u}^{+\infty} e^{-\gamma \frac{x-u}{c-\alpha}} \frac{b-u}{c} k_n\left(\frac{z-b}{c-\alpha} + \frac{b-u}{c}\right) \int_{0}^{z} \Phi_{2n}(x-y)dF(y)
\]
\[
+ \int_{x}^{x-b} \Phi_{1n}(x-y)dF(y) + \int_{x}^{x+\frac{\xi}{\delta}} \Phi_{3n}(x-y)dF(y)
\]
\[
+ \int_{x+\frac{\xi}{\delta}}^{+\infty} u(x, y-x)dF(y) \right] \frac{1}{c-\alpha} dx.
\]

Moreover, first changing variable \( x = h_0(t, u) = u e^{\delta t} + \frac{c(e^{\delta t} - 1)}{\delta} \) in the integrals with respect to \( t \), from \( t \) to \( t_0(t_0 = t_0(u)) \) is the solution of \( h_0(t, u) = u e^{\delta t} + \frac{c(e^{\delta t} - 1)}{\delta} = 0 \) in \( \Phi_{3n}(u) \), and then changing variables \( z = c(t-t_0) \) in the integrals with respect to \( t \) from \( t \) to \( \infty \) in \( \Phi_{3n} \), we obtain
\[
\Phi_{3n}(u) = \int_{u}^{0} (c+\delta u)^\gamma \left(c+\delta x\right)^{-\frac{\gamma}{\delta}} k_n\left(\frac{1}{\delta} \ln \frac{c+\delta x}{c+\delta u}\right) \int_{0}^{x+\frac{\xi}{\delta}} \Phi_{3n}(x-y)dF(y)
\]
\[
+ \int_{x+\frac{\xi}{\delta}}^{+\infty} u(x, y-x)dF(y) dt + \frac{1}{c} \int_{0}^{+\infty} (1 + \frac{\delta}{c u})^\gamma \frac{c}{\delta} e^{-\frac{\gamma}{\delta} z} k_n
\]
\[
\left(\frac{z}{c} + \frac{1}{\delta} \ln \frac{1}{1 + \frac{x}{c}}\right) dz \left[ \int_0^z \Phi_{1n}(x-y) dF(y) + \int_z^{z+\frac{\delta}{\delta}} \Phi_{3n}(x-y) dF(y) \right] + \int_{z+\frac{\delta}{\delta}}^{+\infty} w(z, \ y-z) dF(y) dx.
\]

Then differentiating both sides of \( \Phi_{1n}(u), \Phi_{2n}(u), \) and \( \Phi_{3n}(u) \) with respect to \( u \) and use the relation \( k'_n = -\beta_n k_n + \beta_n k_{n-1} \), we obtain the integro-differential equations:

\[
(1 + \frac{\gamma}{\beta_n}) I - \frac{c}{\beta_n} \frac{\partial}{\partial u} \Phi_{1n}(u) = \Phi_{1n-1}(u), \quad (2.7)
\]

\[
\Phi_1(u) = \int_0^u \Phi_1(u-z) dF(z) + \int_u^{u+\frac{\delta}{\delta}} \Phi_{3n}(u-z) dF(z) + \int_{u+\frac{\delta}{\delta}}^{+\infty} w(u, \ z-u) dF(z). \quad (2.8)
\]

Using (2.7) and (2.8), we have

\[
\prod_{i=1}^n \left[ (1 + \frac{\gamma}{\beta_i}) I - \frac{c}{\beta_i} \frac{\partial}{\partial u} \right] \Phi_1(u) = \int_0^u \Phi_1(u-z) dF(z) + \int_{u+\frac{\delta}{\delta}}^{+\infty} w(u, \ z-u) dF(z) + A(u).
\]

Similarly, we have

\[
\prod_{i=1}^n \left[ (1 + \frac{\gamma}{\beta_i}) I - \frac{c}{\beta_i} \frac{\partial}{\partial u} \right] \Phi_2(u) = \int_{u-b}^u \Phi_1(u-z) dF(z) + \int_{u+\frac{\delta}{\delta}}^{+\infty} \Phi_3(u-z) dF(z) + A(u),
\]

\[
\prod_{i=1}^n \left[ (1 + \frac{\gamma}{\beta_i}) I - \frac{c}{\beta_i} \frac{\partial}{\partial u} \right] \Phi_3(u) = \int_0^{u+\frac{\delta}{\delta}} \Phi_3(u-z) dF(z) + A(u).
\]
The boundary conditions (2.4)-(2.6) can be obtained by using the same arguments as that in [6, 10].

This completes the proof. ■

Corollary 2.2. The absolute ruin probability $\Psi(u)$ satisfies the following integro-differential equations:

When $0 < u < b$, we have

$$\prod_{i=1}^{n} \left[ (I - \frac{c}{\beta_i} \frac{\partial}{\partial u} ) \right] \Psi_1(u) = \int_0^u \Psi_1(u - z) dF(z) + \int_u^{u+\frac{\xi}{\delta}} \Psi_3(u - z) dF(z)$$

$$+ \bar{F}(u + \frac{c}{\delta}), \quad (2.9)$$

when $u > b$, we have

$$\prod_{i=1}^{n} \left[ (I - \frac{c - \alpha_i}{\beta_i} \frac{\partial}{\partial u} ) \right] \Psi_2(u) = \int_{u-b}^u \Psi_1(u - z) dF(z) + \int_0^{u-b} \Psi_2(u - z) dF(z)$$

$$+ \int_u^{u+\frac{\xi}{\delta}} \Psi_3(u - z) dF(z) + \bar{F}(u + \frac{c}{\delta}), \quad (2.10)$$

and when $-\frac{c}{\delta} < u < 0$,

$$\prod_{i=1}^{n} \left[ (I - \frac{c + \delta u}{\beta_i} \frac{\partial}{\partial u} ) \right] \Psi_3(u) = \int_0^{u+\frac{\xi}{\delta}} \Psi_3(u - z) dF(z) + \bar{F}(u + \frac{c}{\delta}), \quad (2.11)$$

with the boundary conditions for $k = 0, 1, 2, \cdots, n - 1$,

$$\left. \prod_{i=1}^{k} \left[ (1 + \frac{\gamma_i}{\beta_i} ) I - \frac{c + \delta u}{\beta_i} \frac{\partial}{\partial u} \right] \Psi_3(u) \right|_{u=-\frac{c}{\delta}+0} = 1, \quad (2.12)$$

$$\left. \prod_{i=1}^{k} \left[ I - \frac{c}{\beta_i} \frac{\partial}{\partial u} \right] \Psi_1(u) \right|_{u=-b} = \left. \prod_{i=1}^{k} \left[ I - \frac{c - \alpha_i}{\beta_i} \frac{\partial}{\partial u} \right] \Psi_2(u) \right|_{u=b^+}, \quad (2.13)$$
\[
\left[ \prod_{i=1}^{k} \left[ I - \frac{c}{\beta_i} \frac{\partial}{\partial u} \right] \right] \Psi_1(u)_{|u=0^+} = \left[ \prod_{i=1}^{k} \left[ I - \frac{c + \delta u}{\beta_i} \frac{\partial}{\partial u} \right] \right] \Psi_3(u)_{|u=\frac{c}{\delta}+0}. \tag{2.14}
\]

**Proof.** If we let \( \gamma = 0 \), and \( \omega = 1 \), then \( A(u) = \mathcal{F}(u + \frac{c}{\delta}) \) and the results follows. Condition (2.12) is obvious: if \( u \downarrow -c/\delta \), then \( A(-c/\delta) = \mathcal{F}(0) = 1 \), which together with (2.4) yields (2.12). If we substitute \( \Psi \) for \( \Phi \) in (2.5) and (2.6), we get the (2.13) and (2.14).

**Corollary 2.3.** Consider the case when \( \omega = 1 \) and \( F(x) = 1 - e^{-\beta x} \) \( (x > 0) \), where \( \beta > 0 \) is a constant, then \( \Phi \) satisfies the differential equations:

When \( 0 < u < b \), we have

\[
(\beta I + \frac{\partial}{\partial u}) \left[ \prod_{i=1}^{n} \left[ (1 + \frac{\gamma}{\beta_i})I - \frac{c}{\beta_i} \frac{\partial}{\partial u} \right] \right] \Phi_1(u) = \beta \Phi_1(u), \tag{2.15}
\]

when \( u > b \), we have

\[
(\beta I + \frac{\partial}{\partial u}) \left[ \prod_{i=1}^{n} \left[ (1 + \frac{\gamma}{\beta_i})I - \frac{c - \alpha}{\beta_i} \frac{\partial}{\partial u} \right] \right] \Phi_2(u) = \beta \Phi_2(u), \tag{2.16}
\]

and when \( -\frac{c}{\delta} < u < 0 \),

\[
(\beta I + \frac{\partial}{\partial u}) \left[ \prod_{i=1}^{n} \left[ (1 + \frac{\gamma}{\beta_i})I - \frac{c - \delta u}{\beta_i} \frac{\partial}{\partial u} \right] \right] \Phi_3(u) = \beta \Phi_3(u) + \sum_{i=1}^{n} \left[ \prod_{j=1, j \neq i}^{n} \left[ (1 + \frac{\gamma}{\beta_j})I - \frac{c - \delta u}{\beta_j} \frac{\partial}{\partial u} \right] \right] \frac{c}{\beta_i} \frac{\partial}{\partial u} \Phi_3(u). \tag{2.17}
\]

with the boundary conditions for \( k = 0, 1, 2, \ldots, n - 1 \),

\[
\left[ \prod_{i=1}^{k} \left[ (1 + \frac{\gamma}{\beta_i})I - \frac{c + \delta u}{\beta_i} \frac{\partial}{\partial u} \right] \right] \Phi_3(u)_{|u=\frac{c}{\delta}+0} = 1, \tag{2.18}
\]
\[
\left[ \prod_{i=1}^{k} \left[ (1 + \frac{\gamma}{\beta_i}) I - \frac{c}{\beta_i} \frac{\partial}{\partial u} \right] \right] \Phi_i(u)|_{u=b^+} = \left[ \prod_{i=1}^{k} \left[ (1 + \frac{\gamma}{\beta_i}) I - \frac{c}{\beta_i} \frac{\partial}{\partial u} \right] \right] \Phi_i(u)|_{u=b^+},
\]
\[
(2.19)
\]

\[
\left[ \prod_{i=1}^{k} \left[ (1 + \frac{\gamma}{\beta_i}) I - \frac{c}{\beta_i} \frac{\partial}{\partial u} \right] \right] \Phi_i(u)|_{u=0^+} = \left[ \prod_{i=1}^{k} \left[ (1 + \frac{\gamma}{\beta_i}) I - \frac{c}{\beta_i} \frac{\partial}{\partial u} \right] \right] \Phi_i(u)|_{u=0^+}.
\]
\[
(2.20)
\]

**Proof.** Applying the operator \((\frac{\partial}{\partial u} + \beta I)\) to (2.1) and noting that

\[
\left( \frac{\partial}{\partial u} + \beta I \right) \int_0^u \Phi_i(u - z) \beta e^{-\beta z} dz = \beta \Phi_i(z),
\]

\[
\left( \frac{\partial}{\partial u} + \beta I \right) \int_u^{u+\xi} \Phi_i(u - z) \beta e^{-\beta z} dz = \beta \Phi_i(z),
\]

\[A'(u) = \beta A(u),\]

we obtain (2.15). Equations (2.16) and (2.17) are proven analogously. The proof of boundary conditions are similarly to that of Theorem 2.1.

### 3. Moment-Generating Function

Since \(M(u, y, b)\) has different paths for \(-c/\delta < u < 0, 0 \leq u \leq b,\)
and \(u > b,\) we define

\[
M(u, y, b) = \begin{cases}
M_1(u, y, b), & 0 \leq u \leq b; \\
M_2(u, y, b), & b < u; \\
M_3(u, y, b), & -\frac{\xi}{\delta} \leq u < 0.
\end{cases}
\]

We denote the moment of \(D_{u,b}\) by \(V_m(u, b) = E[D_{u,b}^m](m \in \mathbb{N})\). Similar to \(M(u, y, b)\), \(V_m(u, b)\) also has different paths for \(-c/\delta < u < 0, 0 \leq u \leq b,\)
and \(u > b,\) we define
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\[ V_m(u, b) = \begin{cases} 
V_{m1}(u, b), & 0 \leq u \leq b; \\
V_{m2}(u, b), & b < u; \\
V_{m3}(u, b), & -\delta \leq u < 0.
\end{cases} \]

In particular, \( V_1(u, b) \) is the expectation of \( D_{u,b} \), that is, \( V_1(u, b) = V(u, b) = E[D_{u,b}] \).

**Theorem 3.1.** When \( 0 < u < b \), we have

\[
\prod_{i=1}^{n} \left( \frac{\lambda y}{\beta_i} \frac{\partial}{\partial y} - \frac{c}{\beta_i} \frac{\partial}{\partial u} + I \right) M_1(u, y, b) = \int_0^u M_1(u - z, y, b) dF(z) \\
+ \int_u^{u+\frac{c}{\delta}} M_3(u - z, y, b) dF(z) + F(u + \frac{c}{\delta}), \quad (3.1)
\]

when \( u > b \), we have

\[
\prod_{i=1}^{n} \left( \frac{\lambda y}{\beta_i} \frac{\partial}{\partial y} - \frac{c - \alpha}{\beta_i} \frac{\partial}{\partial u} + I \right) M_2(u, y, b) = \int_{u-b}^u M_1(u - z, y, b) dF(z) \\
+ \int_0^{u-b} M_2(u - z, y, b) dF(z) + \int_u^{u+\frac{c}{\delta}} M_3(u - z, y, b) dF(z) + F(u + \frac{c}{\delta}), \quad (3.2)
\]

when \( -\frac{c}{\delta} < u < 0 \), we have

\[
\prod_{i=1}^{n} \left( \frac{\lambda y}{\beta_i} \frac{\partial}{\partial y} - \frac{c + \delta u}{\beta_i} \frac{\partial}{\partial u} + I \right) M_3(u, y, b) = \int_0^{u+\frac{c}{\delta}} M_3(u - z, y, b) dF(z) \\
+ F(u + \frac{c}{\delta}), \quad (3.3)
\]

with the boundary conditions for \( k = 0, 1, 2, \ldots, n - 1 \),

\[
\prod_{i=1}^{k} \left( \frac{\lambda y}{\beta_i} \frac{\partial}{\partial y} - \frac{c}{\beta_i} \frac{\partial}{\partial u} + I \right) M_1(u, y, b) \bigg|_{u=b_-} 
\]
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\[
M_2(u, y, b)_{|u=b^+}, \quad (3.4)
\]

\[
\left[ \prod_{i=1}^{k} \left( \frac{\lambda y \partial}{\beta_i \partial y} - \frac{c - \alpha}{\beta_i \partial u} + I \right) \right] M_1(u, y, b)_{|u=0^+},
\]

\[
= \left[ \prod_{i=1}^{k} \left( \frac{\lambda y \partial}{\beta_i \partial y} - \frac{c + \delta u}{\beta_i \partial u} + I \right) \right] M_1(u, y, b)_{|u=0^-}. \quad (3.5)
\]

\[
\left[ \prod_{i=1}^{k} \left( \frac{\lambda y \partial}{\beta_i \partial y} - \frac{c + \delta u}{\beta_i \partial u} + I \right) \right] M_3(u, y, b)_{|u=-\frac{c}{\delta}} = 1. \quad (3.6)
\]

**Proof.** (1) When \( 0 < u < b \), we denote \( M_{1,i}(u, y, b) = M_1(u, y, b|S_{i-1} = t) \) and \( M_1(u, y, b) = M_{1,1}(u, y, b) \), for \( i = 1, 2, \cdots, n \), where \( S_0 = 0, S_i = V_1 + \cdots + V_i \). Since \( \{V_i : i = 1, 2, \cdots, n\} \) are exponentially distributed r.v's, we consider the infinitesimal interval \([S_{i-1}, S_{i-1} + dt]\). By conditioning on the time and amount of the first claim, and whether the claim causes absolute ruin, we obtain

\[
M_{1,i}(u, y, b) = (1 - \beta_i dt) M_{1,i}(u + cdt, ye^{-\lambda dt}, b) + \beta_i dt M_{1,i+1}
\]

\[
\times (u + cdt, ye^{-\lambda dt}, b) + o(dt). \quad (3.7)
\]

Taylor’s expansion gives

\[
M_{1,i}(u + cdt, ye^{-\lambda dt}, b) = M_{1,i}(u, y, b) + c \frac{\partial M_{1,i}}{\partial u}(u, y, b)
\]

\[
- y\lambda \frac{\partial M_{1,i}}{\partial y} (u, y, b) dt + o(dt). \quad (3.8)
\]

Substituting (3.8) into (3.7), dividing both sides of (3.7) by \( t \), letting \( t \to \infty \) and rearranging, we obtain

\[
c \frac{\partial M_{1,i}}{\partial u}(u, y, b) - \beta_i M_{1,i}(u, y, b) - y\lambda \frac{\partial M_{1,i}}{\partial y} (u, y, b) + \beta_i M_{1,i+1}(u, y, b) = 0.
\]
That is,
\[
M_{1,i+1}(u, y, b) = \frac{\lambda y \frac{\partial}{\partial y} M_{1,i}(u, y, b) + \beta_1 I - c \frac{\partial}{\partial u} M_{1,i}(u, y, b)}{\beta_i} M_{1,i}(u, y, b),
\]
where \(i = 1, 2, \ldots, n-1\). Now let \(i = n\), we have
\[
M_{1,n}(u, y, b) = (1 - \beta_n dt)M_{1,n}(u + cdt, ye^{-\lambda dt}, b)
\]
\[
+ \beta_n dt \int_0^u M_1(u - z, y, b)dF(z) + \int_u^{u + \frac{c}{\delta}} M_3(u - z, y, b)dF(z)
\]
\[
- \bar{F}(u + \frac{c}{\delta}) + o(dt).
\]
That is,
\[
\frac{\lambda y \frac{\partial}{\partial y} + \beta_n I - c \frac{\partial}{\partial u}}{\beta_n} M_{1,n}(u, y, b) = \int_0^u M_1(u - v, y, b)dF(v)
\]
\[
+ \int_u^{u + \frac{c}{\delta}} M_3(u - z, y, b)dF(z)
\]
\[
+ \bar{F}(u + \frac{c}{\delta}). \tag{3.10}
\]
Substituting (3.10) into (3.9), we have (3.1).

(2) Using the arguments similar to those used in (3.6) and (3.7), we have (3.2) and (3.3).

According to [6], we have
\[
M_{1,j}(u, y, b)\big|_{u=0} = M_{2,j}(u, y, b)\big|_{u=0},
\]
\(j = 1, 2, \ldots, n\), and
\[
\frac{\partial M_{1,j}(u, y, b)}{\partial u}\bigg|_{u=0} = \frac{\partial M_{2,j}(u, y, b)}{\partial u}\bigg|_{u=0}, \quad j = 1, 2, \ldots, n,
\]
which together with (3.9) yields (3.4). Similarly, we can get the boundary conditions of (3.5) and (3.6).
\\[\blacksquare\]
Theorem 3.2. When $0 < u < b$, 

\[
\left[ \prod_{i=1}^{n} \left( 1 + \frac{m \lambda}{\mu_i} \right) \right] V_{m1}(u) = \int_{0}^{u} V_{m1}(u - y) dF(y) + \int_{u}^{u+b} V_{m1}(u - y) dF(y), \quad (3.11)
\]

when $u > b$,

\[
\left[ \prod_{i=1}^{n} \left( 1 + \frac{m \lambda}{\mu_i} \right) \right] V_{m2}(u) = \int_{u-b}^{u} V_{m1}(u - y) dF(y) + \int_{0}^{u-b} V_{m2}(u - y) dF(y) + \int_{u}^{u+b} V_{m3}(u - y) dF(y), \quad (3.12)
\]

and when $-\frac{c}{\delta} < u < 0$,

\[
\left[ \prod_{i=1}^{k} \left( \lambda \frac{\partial}{\partial y} - \frac{c - \alpha}{\mu_i} \frac{\partial}{\partial u} + I \right) \right] V_{m3}(u) = \int_{u}^{u+b} V_{m3}(u - y) dF(y), \quad (3.13)
\]

with the boundary conditions for $k = 0, 1, \cdots, n - 1$,

\[
\left[ \prod_{i=1}^{k} \left( \lambda \frac{\partial}{\partial y} - \frac{c + \delta u}{\mu_i} \frac{\partial}{\partial u} + I \right) \right] V_{m1}(u)|_{u=b} = \prod_{i=1}^{k} \left( \lambda \frac{\partial}{\partial y} - \frac{c - \alpha}{\mu_i} \frac{\partial}{\partial u} + I \right) V_{m2}(u)|_{u=b},
\]

\[
\left[ \prod_{i=1}^{k} \left( \lambda \frac{\partial}{\partial y} - \frac{c + \delta u}{\mu_i} \frac{\partial}{\partial u} + I \right) \right] V_{m1}(u)|_{u=0} = \prod_{i=1}^{k} \left( \lambda \frac{\partial}{\partial y} - \frac{c - \alpha}{\mu_i} \frac{\partial}{\partial u} + I \right) V_{m2}(u)|_{u=0},
\]

\[
\left[ \prod_{i=1}^{k} \left( \lambda \frac{\partial}{\partial y} - \frac{c + \delta u}{\mu_i} \frac{\partial}{\partial u} + I \right) \right] V_{m3}(u)|_{u=-\frac{c}{\delta}} = 0.
\]

Proof. By the definitions of $M(u, y, b)$ and $V_{m}(u, b)$, we obtain

\[
M(u, y, b) = 1 + \sum_{i=1}^{\infty} \frac{y^m}{m!} V_{m}(u, b). \quad (3.14)
\]
Substituting (3.11) into (3.1), (3.2), and (3.3), respectively, and comparing the coefficients of \( y^m (m \in N) \), yielding the results. ■

**Corollary 3.3.** Consider the case when \( F(x) = 1 - e^{-\beta x} (x > 0) \), then \( V_m(u, b) \) satisfies the differential equations:

When \( 0 \leq u \leq b \),

\[
(\beta I + \frac{\partial}{\partial u}) \left[ \prod_{i=1}^{n} [(1 + \frac{m\lambda_j}{\beta_i}) I - \frac{c}{\beta_i} \frac{\partial}{\partial u}] \right] V_{m1}(u) = \beta V_{m1}(u), \tag{3.15}
\]

when \( u > b \), we have

\[
(\beta I + \frac{\partial}{\partial u}) \left[ \prod_{i=1}^{n} [(1 + \frac{m\lambda_j}{\beta_i}) I - \frac{c - \alpha}{\beta_i} \frac{\partial}{\partial u}] \right] V_{m2}(u) = \beta V_{m2}(u), \tag{3.16}
\]

and when \( -\frac{c}{\delta} \leq u < 0 \),

\[
(\beta I + \frac{\partial}{\partial u}) \left[ \prod_{i=1}^{n} [(1 + \frac{m\lambda_j}{\beta_i}) I - \frac{c + \delta u}{\beta_i} \frac{\partial}{\partial u}] \right] V_{m3}(u)
= \beta V_{m3}(u) + \left( \sum_{j=1, j \neq i}^{n} \left[ \prod_{j=1}^{n} (1 + \frac{m\lambda_j}{\beta_j}) I - \frac{c + \delta u}{\beta_j} \frac{\partial}{\partial u} \right] \right) \beta \frac{\partial}{\partial u} V_{m3}(u), \tag{3.17}
\]

with the boundary conditions for \( k = 0, 1, \ldots, n - 1 \),

\[
\left[ \prod_{i=1}^{k} \left( \frac{\lambda y}{\beta_i} \frac{\partial}{\partial y} - \frac{c + \delta u}{\beta_i} \frac{\partial}{\partial u} + I \right) \right] V_{m1}(u)|_{u=b^+} = \left[ \prod_{i=1}^{k} \left( \frac{\lambda y}{\beta_i} \frac{\partial}{\partial y} - \frac{c - \alpha}{\beta_i} \frac{\partial}{\partial u} + I \right) \right] V_{m2}(u)|_{u=b^+},
\]

\[
\left[ \prod_{i=1}^{k} \left( \frac{\lambda y}{\beta_i} \frac{\partial}{\partial y} - \frac{c + \delta u}{\beta_i} \frac{\partial}{\partial u} + I \right) \right] V_{m1}(u)|_{u=0^-} = \left[ \prod_{i=1}^{k} \left( \frac{\lambda y}{\beta_i} \frac{\partial}{\partial y} - \frac{c - \alpha}{\beta_i} \frac{\partial}{\partial u} + I \right) \right] V_{m3}(u)|_{u=0^-},
\]

\[
\left[ \prod_{i=1}^{k} \left( \frac{\lambda y}{\beta_i} \frac{\partial}{\partial y} - \frac{c + \delta u}{\beta_i} \frac{\partial}{\partial u} + I \right) \right] V_{m3}(u)|_{u=-\frac{c}{\delta}} = 0.
\]

**Proof.** The proof is similar to that of Corollary 2.3.
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References